

Confinement and mass gap in abelian gauge

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Received: 27 November 2002 /

Published online: 14 April 2003 – © Springer-Verlag / Società Italiana di Fisica 2003

Abstract. First, we present a simple confining abelian pure gauge theory. Classically, its kinetic term is not positive definite, and it contains a simple UV regularized F^4 interaction. This provokes the formation of a condensate $\phi \sim F^2$ such that, at the saddle point $\hat{\phi}$ of the effective potential, the wave function normalization constant of the abelian gauge fields $Z_{\text{eff}}(\hat{\phi})$ vanishes exactly. Then we study $SU(2)$ pure Yang–Mills theory in an abelian gauge and introduce an auxiliary field ρ for a BRST invariant condensate of dimension 2, which renders the charged sector massive. Under simple assumptions its effective low energy theory reduces to the confining abelian model discussed before, and the VEV of ρ is seen to scale correctly with the renormalization point. Under these assumptions, the confinement condition $Z_{\text{eff}} = 0$ also holds for the massive charged sector, which suppresses the couplings of the charged fields to the abelian gauge bosons in the infrared regime.

1 Introduction

Various aspects of the confining phase of Yang–Mills theories become more transparent in the abelian gauge [1, 2], notably the phenomenon of monopole condensation [1, 3] according to which the vacuum behaves as a dual superconductor. Since these monopoles are essentially configurations of the gauge fields belonging to the $U(1)$ Heisenberg sub-algebras of $SU(N)$, the abelian subsector of non-abelian gauge theories plays the dominant role for this mechanism responsible for confinement. This phenomenon is called abelian dominance [4, 5].

In the abelian gauge the dynamics of the “abelian” gauge fields is thus expected to differ considerably from the dynamics of the “charged” gauge fields (associated to off-diagonal generators, and charged with respect to at least one $U(1)$ subgroup). Whereas the abelian gauge fields are expected to reproduce essentially the phenomenon of monopole condensation of compact QED in the confining phase [6], the charged gauge fields are expected to be massive and contribute only sub-dominantly to large distance phenomena. This massive behavior of the charged gauge field propagators has been observed in lattice studies [5].

As dynamical origin of the masses of the charged gauge fields ghost–antighost condensates [7, 8] and bi-ghost condensates [9] of dimension 2 have been proposed. Notably a

particular combination of ghost–antighost and gauge field condensates is BRST invariant (up to a total derivative) both in the abelian gauge and a generalization of the Lorentz gauge [10]. If this particular condensate is realized, it describes simultaneously the dimension 2 gauge field condensate discussed independently in [11, 12] in the Landau gauge. Note, however, that the ghost–antighost condensates in [7] do not allow for such a BRST invariant extension and thus necessarily induce a spontaneous breakdown of BRST symmetry.

In [7–9] the formation of the ghost condensate has been related to the presence of four-ghost interactions in the corresponding gauges. From the Nambu–Jona–Lasinio model [13] it is well known that four-Fermi interactions can provoke the formation of bilinear condensates. However, here the coefficient of the four-ghost interaction is proportional to an a priori arbitrary gauge parameter α . Hence the scale of the condensate is not given by the confinement scale Λ_{QCD} (we continue to denote this scale by an index QCD, although we will consider only pure Yang–Mills theories), unless one fine-tunes α to be proportional to the first coefficient of the β function [7–9]. One of the purposes of the present paper is to present a different mechanism for the formation of a dimension 2 condensate, which leads automatically to its proportionality to Λ_{QCD}^2 .

Moreover, one would like to learn more about the relation between the dimension 2 condensate and the properties of the confining phase as the area law of the Wilson loop, the condensation of monopoles, and the vanishing of the effective wave function normalization constants Z_{eff}

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[14,15]. (Here Z_{eff} corresponds to Z_3^{-1} in [15], and the relation between the vanishing of Z_{eff} and the Kugo–Ojima criterion for confinement [16] has been discussed in [17].) In the Landau gauge, relations of the dimension 2 condensate with confinement have been discussed in [11,18].

The description of monopole condensation requires either the introduction of the 't Hooft monopole operator [19] or the introduction of an antisymmetric tensor field $B_{\mu\nu}$ [20–23], which is dual to a monopole condensate and couples to the surface of the Wilson loop. In [22] the relation between monopole condensation, the area law of the Wilson loop and $Z_{\text{eff}} = 0$ has been discussed in the formulation of the Yang–Mills partition function involving $B_{\mu\nu}$, and in [24] these relations have been shown to hold in a solvable abelian model in the large N limit.

In the present paper we will not introduce a $B_{\mu\nu}$ field, and concentrate on $Z_{\text{eff}} = 0$ as a criterion for confinement. In the first part of the paper (Sect. 2) we present a simple confining abelian gauge theory, which involves a non-renormalizable interaction $\sim \lambda^2 F^4$ and has to be equipped with a UV cutoff as, e.g., in the form of a decreasing momentum dependent form factor in the interaction term. Also the kinetic term is assumed to show some non-trivial momentum dependence. (Both these features of the abelian model are obtained in Sect. 3, where the abelian model is derived from $SU(2)$ pure Yang–Mills theory in the abelian gauge.)

Then we will introduce a dimension 4 condensate ϕ for the (abelian) field strength squared. We show that the effective potential for this condensate can develop a saddle point, which corresponds exactly to $Z_{\text{eff}}(\phi) = 0$ and where the propagator of the abelian gauge fields behaves like q^{-4} for $q^2 \rightarrow 0$. As in the model in [24] this saddle point is only “visible” if one introduces an infrared cutoff, and studies the limit where the infrared cutoff is removed. We will introduce a momentum space cutoff k^2 ; alternatively the system can be placed in a finite volume, and then the infinite volume limit can be considered. In this limit the confining saddle point turns into an essential singularity of the effective potential which, however, remains finite at this point.

In Sect. 3 we turn to $SU(2)$ pure Yang–Mills theory in the abelian gauge, and introduce an auxiliary field ρ for the above-mentioned condensate of dimension 2, which renders the charged gauge fields (and ghosts) massive. After integrating out these charged fields, the remaining effective action for the abelian gauge field (and ρ) resembles the abelian model of Sect. 2. Repeating the steps of Sect. 2 one finds that the “confining” saddle point of the effective potential now also fixes ρ to be of $\mathcal{O}(A_{\text{QCD}}^2)$.

We will argue that $Z_{\text{eff}} = 0$ for the abelian gauge fields induces also $Z_{\text{eff}} = 0$ for the charged gauge fields (and ghosts), invoking renormalization group arguments (as in [14,15]). This has less dramatic effects on the (massive) propagators of the charged fields, but now the couplings of the charged fields to the abelian gauge fields, which are induced by the $U(1)$ covariant derivatives in the kinetic terms of the charged fields, vanish in the infrared.

Interestingly, the essential features of the mechanism for confinement considered here are visible already in a loop expansion of the effective action, once the corresponding auxiliary fields are introduced, and once certain perturbatively irrelevant terms in the effective action are taken into account.

In Sect. 4 we conclude, summarizing the essential properties of our approach.

2 A confining abelian gauge theory

A class of confining abelian gauge theories has been discussed in [24]. These models involve antisymmetric tensor fields $B_{\mu\nu}$, and are solvable in the $1/N$ limit. In the present section we present a simplified version of these models: first, we do not introduce antisymmetric tensor fields $B_{\mu\nu}$, and second, we confine ourselves to $N = 1$. The field content is thus just an abelian gauge field A_μ . In the absence of a $1/N$ limit the “solution” of the model is no longer quantitatively exact, but its qualitative features remain the same (see the discussion below).

In the presence of antisymmetric tensor fields the area law of the Wilson loop is easily obtained in the confining phase, since antisymmetric tensor fields couple naturally to the enclosed surface. In the formulation with abelian gauge fields only, the criterion for confinement becomes a q^{-4} behavior of its propagator in the infrared limit, which implies a vanishing wave function renormalization constant Z_{eff} as in [14,15]. (The relation between the q^{-4} behavior of the gauge field propagator and the area law of the Wilson loop has been discussed in [25].)

The simplest confining abelian gauge model involves just a kinetic term including higher derivatives, $(1/4) F_{\mu\nu} Z^A (-\square) F_{\mu\nu}$, and a $\lambda^2 F^4$ interaction. For the model to be confining, Z^A and the dimensional coupling λ^2 have to satisfy some inequality (see below); notably $Z^A(0)$ has to be negative. Clearly this model is “non-renormalizable”, and has to be supplemented with an UV cutoff Λ . This makes sense, since it is only believed to correspond to an “effective low energy theory” of a non-abelian gauge theory in the abelian gauge, where the off-diagonal gauge fields are massive.

We will implement an UV cutoff by supplementing the $\lambda^2 F^4$ interaction with a momentum dependent form factor, which decreases sufficiently rapidly at large momenta. Again this form of the UV cutoff is motivated by the idea that the $\lambda^2 F^4$ interaction is induced by loops of massive non-abelian gauge fields; hence the UV cutoff is naturally of the order of the non-abelian gauge field masses. Actually, in the Yang–Mills case the corresponding decay of the induced form factor is not sufficiently rapid in order to prevent logarithmic divergences, which require the standard counter terms of Yang–Mills theories. Since we are interested in the infrared behavior of the present model, however, we will simplify the treatment of its UV behavior and replace the “soft” UV cutoff by a “sharp” UV cutoff.

$$e^{-G(J)} = \int \mathcal{D}A \mathcal{D}\phi e^{-\int d^4x \left\{ (1/4)F_{\mu\nu}Z^A(-\square)F_{\mu\nu} - (1/8)\phi^2(x) + (\lambda/4)\phi(x)\mathcal{O}(x) + (1/2\beta)(\partial_\mu A_\mu)^2 - J_\mu A_\mu \right\}} \quad (2.4)$$

Thus we take as action of the model (including a standard gauge fixing term)

$$S(A_\mu) = \int d^4x \left\{ \frac{1}{4}F_{\mu\nu}Z^A(-\square)F_{\mu\nu} + \frac{\lambda^2}{8}\mathcal{O}(x)\mathcal{O}(x) + \frac{1}{2\beta}(\partial_\mu A_\mu)^2 \right\}, \quad (2.1)$$

with

$$\mathcal{O}(x) = \int \mathcal{D}q e^{iqx} \int \mathcal{D}p \theta(\Lambda^2 - p^2) \times F_{\mu\nu}(p+q)F_{\mu\nu}(q-p), \quad (2.2)$$

where $\mathcal{D}q \equiv d^4q/(2\pi)^4$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ or its Fourier transform. The θ -function introduced in (2.2) suffices to regularize all UV divergences in the approximation considered below. For $Z^A(q^2)$ we make the choice

$$Z^A(q^2) = Z_0^A + \frac{a_1 q^2}{a_2 \Lambda^2 + q^2} \quad (2.3)$$

with $Z_0^A + a_1 > 0$ such that $Z^A(q^2 \rightarrow \infty) > 0$, but later we will allow for $Z^A(0) = Z_0^A < 0$. Again this choice will be motivated in the next section by the idea that $S(A_\mu)$ in (2.1) corresponds to an effective low energy theory. The constants a_1 and a_2 are assumed to be positive and of $\mathcal{O}(1)$.

Next we rewrite the (Euclidean) partition function of the model, introducing an auxiliary field $\phi(x)$ for the operator (2.2): (see (2.4) on top of the page).

The coefficient of the ϕ^2 term in the exponent in (2.4) seems to have the “wrong” sign. However, the Gaussian path integral over $\phi(x)$ is still well defined by analytic continuation and gives back the original action (2.1); corresponding procedures for auxiliary fields with “wrong” sign quadratic terms are well known from, e.g., supersymmetric theories in the formulation with auxiliary fields F and D .

Now the A_μ path integral is Gaussian; the terms quadratic in A_μ can be written as (up to the gauge fixing term, and for constant ϕ for simplicity)

$$\frac{1}{4} \int \mathcal{D}q F_{\mu\nu}(-q) Z_{\text{eff}}^A(\phi, q^2) F_{\mu\nu}(q), \quad (2.5)$$

with

$$Z_{\text{eff}}^A(\phi, q^2) = Z_0^A + \frac{a_1 q^2}{a_2 \Lambda^2 + q^2} + \lambda \phi \theta(\Lambda^2 - q^2). \quad (2.6)$$

As usual we allow ourselves to interchange the A_μ and ϕ path integrals in (2.4). The logarithm of the determinant of the Gaussian A_μ path integral then contributes to the effective potential $V_{\text{eff}}(\phi)$, which has to be used to determine the saddle point of the remaining ϕ path integral.

The relevant point is that the saddle point $\hat{\phi}$ of $V_{\text{eff}}(\phi)$, which represents the confining phase, will correspond precisely to $Z_{\text{eff}}^A(\hat{\phi}, 0) = 0$.

The Coleman–Weinberg contribution of the Gaussian A_μ path integral to $V_{\text{eff}}(\phi)$ reads (in the Landau gauge $\beta = 0$)

$$\Delta V(\phi) = \frac{3}{2} \int \frac{q^2 dq^2}{16\pi^2} \ln(Z_{\text{eff}}^A(\phi, q^2)). \quad (2.7)$$

Note that, due to the θ -function in (2.6), all ϕ -dependent terms in (2.7) are ultraviolet finite; these ϕ -dependent terms remain unchanged by introducing an UV cutoff Λ^2 for the q^2 integral and omitting the θ -function in Z_{eff}^A .

Since $Z_{\text{eff}}^A(\phi, q^2)$ may turn negative for small q^2 and small ϕ , the infrared behavior of the q^2 integral in (2.7) is very delicate. Its correct behavior can only be obtained by

- (i) introducing an infrared cutoff k^2 (for simplicity, we employ a sharp cutoff of the q^2 integral; the final result does not depend, however, on this choice),
- (ii) study the saddle point(s) of $V_{\text{eff}}(\phi)$ for finite k^2 and
- (iii) take the limit $k^2 \rightarrow 0$ at the end.

Hence, instead of (2.7), we write

$$\Delta V(\phi) = \frac{3}{2} \int_{k^2}^{\Lambda^2} \frac{q^2 dq^2}{16\pi^2} \ln(Z_{\text{eff}}^A(\phi, q^2)), \quad (2.8)$$

where now the θ -function on the right hand side of (2.6) is replaced by 1.

The result of the q^2 integral is most easily written in terms of the combination

$$\Sigma(\phi) = \frac{a_2 \Lambda^2 Z_{\text{eff}}^A(\phi, 0)}{a_1 + Z_{\text{eff}}^A(\phi, 0)}, \quad (2.9)$$

where

$$Z_{\text{eff}}^A(\phi, 0) = Z_0^A + \lambda \phi. \quad (2.10)$$

Now the total potential $V(\phi) = -(1/8)\phi^2 + \Delta V(\phi)$ becomes

$$V(\phi) = -\frac{1}{8}\phi^2 + \frac{3}{64\pi^2} \left[(\Sigma^2 - k^4) \ln(\Sigma + k^2) + (\Lambda^4 - \Sigma^2) \ln(\Sigma + \Lambda^2) + \Sigma(\Lambda^2 - k^2) + (\Lambda^4 - k^4) \ln(a_1 + Z_{\text{eff}}^A(\phi, 0)) \right] + (\phi\text{-independent}). \quad (2.11)$$

The saddle point condition then reads

$$0 = \left. \frac{dV(\phi)}{d\phi} \right|_{\hat{\phi}} = -\frac{1}{4}\hat{\phi} + \frac{3a_1 a_2 \lambda \Lambda^2}{32\pi^2 (a_1 + Z_{\text{eff}}^A(\hat{\phi}, 0))^2} \left[\Sigma \ln\left(\frac{\Sigma + k^2}{\Sigma + \Lambda^2}\right) \right]$$

$$+\Lambda^2 - k^2 + \frac{(\Lambda^4 - k^4)(a_1 + Z_{\text{eff}}^A(\widehat{\phi}, 0))}{2a_1a_2\Lambda^2} \Big] . \quad (2.12)$$

In the limit $k^2 \rightarrow 0$ the product of Σ with the logarithm in (2.12) can show the following subtle behavior:

$$\begin{aligned} k^2 &\rightarrow 0 , \\ \Sigma &\rightarrow 0_{-\varepsilon} , \\ \Sigma \ln \left(\frac{\Sigma + k^2}{\Sigma + \Lambda^2} \right) &\rightarrow K = \text{const.} \end{aligned} \quad (2.13)$$

where the constant K is positive and can be chosen such that (2.12) is satisfied for $k^2 \rightarrow 0$, provided

$$\frac{1}{4}\widehat{\phi} - \frac{3a_2\lambda\Lambda^4}{32\pi^2a_1} \left(1 + \frac{1}{2a_2} \right) > 0 , \quad (2.14)$$

which we assume in the following. Note that $\Sigma \rightarrow 0$ corresponds to

$$Z_{\text{eff}}^A(\widehat{\phi}, 0) = Z_0^A + \lambda\widehat{\phi} = 0 , \quad (2.15)$$

which has already been used in order to derive (2.14) from (2.12). Note also that the saddle point (2.13) would be invisible if we would put $k^2 = 0$ from the beginning: at the corresponding value $\widehat{\phi}$ (corresponding to $\Sigma = 0$) the potential $V(\widehat{\phi})|_{k^2=0}$ and its first derivatives are finite, but all higher derivatives diverge. Only after regularization of this singularity (through the infrared cutoff k^2) one finds that this essential singularity of $V(\phi)$ contains a ‘‘hidden’’ saddle point.

Equation (2.15) corresponds to the result announced above: at the confining saddle point (or in the confining phase) the auxiliary field ϕ , which corresponds to a condensate $\langle F_{\mu\nu}F_{\mu\nu} \rangle$, arranges itself such that $Z_{\text{eff}}^A = 0$ exactly (without fine-tuning).

However, the original parameters of the model have to satisfy some inequality for the confining phase to exist: from (2.14) and (2.15) one easily finds

$$Z_0^A < -\frac{3a_2\lambda^2\Lambda^4}{8\pi^2a_1} \left(1 + \frac{1}{2a_2} \right) , \quad (2.16)$$

i.e. notably

$$Z_0^A < 0 \quad (2.17)$$

for $a_1, a_2 > 0$, which we do assume. Equation (2.17) explains the formation of the condensate $\phi \sim \langle F_{\mu\nu}F_{\mu\nu} \rangle$: now the action (2.1) is unstable at the origin of constant modes of $F_{\mu\nu}$ already classically (the classical A_μ propagator, for $\phi = 0$, would be tachyonic for $q^2 \rightarrow 0$).

The remarkable point is, however, that the condensate $\widehat{\phi}$ arranges itself in the confining phase such that the A_μ propagator in the background $\widehat{\phi}$ shows a q^{-4} behavior for $q^2 \rightarrow 0$ (which is related, of course, to $Z_{\text{eff}}^A(\widehat{\phi}) = 0$): after replacing ϕ by $\widehat{\phi}$ in the exponent of the partition function (2.4), i.e. after approximating the ϕ path integral by its saddle point, the A_μ propagator can be obtained from

the inverse of $(1/2)\delta^2 G(J)/\delta J_\mu(-q)\delta J_\nu(q)$. In the Landau gauge, $\beta \rightarrow 0$, one finds

$$P_{\mu\nu}^A = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{a_2\Lambda^2 + q^2}{a_1q^4} , \quad (2.18)$$

which coincides with the expression for $P_{\mu\nu}^A$ in confining models with antisymmetric tensor fields $B_{\mu\nu}$ [22, 24].

The saddle point approximation for the ϕ path integral can be rendered exact within a $1/N$ expansion [24], i.e. after replacing A_μ by A_μ^a , $a = 1, \dots, N$, and rescaling the coupling correspondingly. In the present case the ϕ path integral is, in principle, not trivial. Note, however, that $d^2V(\phi)/d\phi^2|_{\widehat{\phi}} = -\infty$, i.e. the ϕ propagator vanishes in the confining phase at vanishing momentum. Also, the coupling of ϕ to $F_{\mu\nu}F_{\mu\nu}$ is equipped with an UV regulator (form factor), hence perturbation theory in powers of this coupling has a good chance to converge rapidly. A detailed study of this problem is, however, beyond the scope of the present paper.

3 Mass gap and confinement in $SU(2)$ Yang–Mills theory

As stated in the introduction, we consider pure $SU(2)$ Euclidean Yang–Mills theory in a (continuum version of) the (maximal) abelian gauge [26–28]. A_μ denotes the abelian gauge field associated to the $U(1)$ subgroup, and W_μ^\pm the remaining charged gauge fields. The classical action reads

$$S = \int d^4x \{ \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} \} \quad (3.1)$$

where \mathcal{L}_{YM} is the Yang–Mills Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{YM}} &= \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\ &+ \frac{1}{2} (D_\mu W_\nu^+ - D_\nu W_\mu^+) (D_\mu W_\nu^- - D_\nu W_\mu^-) \\ &+ \frac{ig}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+) \\ &- \frac{g^2}{4} (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+)^2 . \end{aligned} \quad (3.2)$$

Here the D_μ denote the $U(1)$ covariant derivatives $\partial_\mu \pm igA_\mu$. After elimination of the Nakanishi–Lautrup auxiliary fields the gauge fixing part \mathcal{L}_{GF} reads

$$\begin{aligned} \mathcal{L}_{\text{GF}} &= \frac{1}{2\beta} (\partial_\mu A_\mu)^2 + \frac{1}{\alpha} D_\mu W_\mu^+ D_\nu W_\nu^- \\ &+ \partial_\mu \bar{c}^3 (\partial_\mu c^3 + ig(W_\mu^+ c^- - W_\mu^- c^+)) \\ &+ D_\mu \bar{c}^+ D_\mu c^- + D_\mu \bar{c}^- D_\mu c^+ \\ &+ g^2 (W_\mu^+ \bar{c}^- - W_\mu^- \bar{c}^+) (W_\mu^+ \bar{c}^- - W_\mu^- \bar{c}^+) \\ &- \alpha g^2 \bar{c}^+ c^- \bar{c}^- c^+ . \end{aligned} \quad (3.3)$$

The neutral ghosts c^3, \bar{c}^3 actually decouple and will play no role in the following. (There are no vertices involving c^3 .)

Now we introduce an auxiliary field ρ for the bilinear dimension 2 condensate

$$W_\mu^+ W_\mu^- + \alpha (\bar{c}^+ c^- + \bar{c}^- c^+) . \quad (3.4)$$

Under BRST transformations this operator transforms into the total derivative $\partial_\mu (W_\mu^+ c^- + W_\mu^- c^+)$ [10]; for the explicit BRST transformations corresponding to the conventions implicit in \mathcal{L}_{GF} see [28].

The introduction of ρ corresponds to adding to S the complete square

$$\mathcal{L}_\rho = \frac{1}{2g^2} (\rho + g^2 W_\mu^+ W_\mu^- + \alpha g^2 (\bar{c}^+ c^- + \bar{c}^- c^+))^2 , \quad (3.5)$$

and of course ρ has to transform under BRST transformations in the same way as the negative of the operator (3.4). It is understood that now a ρ path integral has to be performed.

Note that, when adding (3.5) to S , we made no effort to cancel the quartic ghost interaction term in (3.3) as in [7–9]; the powers of g^2 in (3.5) have just been introduced in order to facilitate their bookkeeping. It is of course true that, once this term has been cancelled, S is quadratic in the charged ghosts, and the ghost path integral can be performed trivially. We do not believe, however, that the resulting contribution to the effective potential of ρ is dominant and fixes its VEV. We will identify another contribution below, which is more relevant for a small enough gauge parameter α (but still $\alpha \sim \mathcal{O}(1)$). In any case the absence of the four-ghost interaction is no scale invariant statement, since it is re-generated by W_μ^\pm -loops (which are not $1/N$ -suppressed, as in solvable four-Fermi models). Also the physical consequences of an auxiliary field as introduced in (3.5) should be independent from the conventions chosen for the corresponding coefficients; in (3.5) we choose, for simplicity, conventions such that the induced mass terms \mathcal{L}_m for the charged fields are simply expressed in terms of ρ :

$$\mathcal{L}_m = \rho W_\mu^+ W_\mu^- + \alpha \rho (\bar{c}^+ c^- + \bar{c}^- c^+) . \quad (3.6)$$

Next we wish to integrate over the charged fields W_μ^\pm , c^\pm . However, in order to control the UV divergences, one should integrate simultaneously over the high momentum modes of the abelian field $A_\mu(p^2)$ with, say, $p^2 > \Lambda^2$. Of course, it is not trivial to implement such an intermediate scale in a gauge (or BRST) invariant way. The best one can do is to implement the constraint $p^2 > \Lambda^2$ in a Wilsonian sense (i.e. by modifying the A_μ propagators correspondingly) which allows one to control the BRST symmetry with the help of modified Slavnov–Taylor identities [29]. For our subsequent qualitative results and its essential features the details of this procedure will play no role, however. After having renormalized the UV divergences by, e.g., dimensional regularization, we are left with an induced effective action $\Gamma_{\text{eff}}(A_\mu, \rho)$, which is a functional of the low momentum modes of A_μ and of ρ .

Thus we rewrite the full Yang–Mills path integral – including the path integral over ρ – as

$$\int \mathcal{D}A \mathcal{D}W \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\rho e^{-\int d^4x \{ \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_\rho \}}$$

$$= \int \mathcal{D}A_{<\Lambda^2} \mathcal{D}\rho e^{-\Gamma_{\text{eff}}(A, \rho)} , \quad (3.7)$$

where the index $<\Lambda^2$ attached to $\mathcal{D}A$ denotes the restriction to modes with $p^2 \lesssim \Lambda^2$, and where a $U(1)$ gauge fixing term (the first term in (3.3)) is understood in Γ_{eff} .

Let us first have a look at the term quadratic in A_μ in $\Gamma_{\text{eff}}(A, \rho)$. Due to the $U(1)$ gauge invariance it has to be of the form

$$\int \mathcal{D}q \frac{1}{4} F_{\mu\nu}(-q) (Z_0^A(\rho, \mu^2) + f^A(q^2, \rho)) F_{\mu\nu}(q) . \quad (3.8)$$

Here we have suppressed the dependence on the gauge parameter α which we assume to be of $\mathcal{O}(1)$ subsequently such that, from (3.6), the masses of all charged fields are of $\mathcal{O}(\rho)$. Of course $Z_0^A(\rho, \mu^2)$ is of the form $Z_0^A(\rho, \mu^2) = 1 +$ loop corrections, and we define the splitting between $Z_0^A(\rho, \mu^2)$ and the q^2 -dependence parameterized by $f^A(q^2, \rho)$ such that $f^A(0, \rho) = 0$.

Next we discuss some particular features of the scale anomaly in abelian gauges. A natural choice for the running gauge coupling g_R (but not necessarily a physical one; see below) is the coupling g in the $U(1)$ covariant derivative $D_\mu = \partial_\mu \pm igA_\mu$, after rescaling A_μ such that its kinetic term (3.8) is properly normalized (at $q^2 = 0$). In the case of (3.8) this immediately leads to

$$g_R^2 = \frac{g^2}{Z_0^A(\rho, \mu^2)} , \quad (3.9)$$

where g^2 is constant. The derivative of g_R^2 with respect to its dimensionful arguments gives the β -function in a herewith defined renormalization scheme. In our case one finds by inspecting the diagrams which contribute to Z_0^A , and taking into account that the circulating charged fields have masses given by ρ , that to one-loop order $Z_0^{A(1)}$ is independent of the infrared cutoff Λ^2 of the abelian gauge fields, since no internal A_μ propagators appear. Thus $Z_0^{A(1)}(\rho, \mu^2)$ depends on ρ as dictated by the universal one-loop coefficient β_0 of the β -function (cf. [27]):

$$Z_0^{A(1)}(\rho, \mu^2) = 1 - g^2 \beta_0 \ln \left(\frac{\mu^2}{c_1 \rho} \right) + \mathcal{O}(g^4) \quad \text{with} \\ \beta_0 = \frac{11}{24\pi^2} , \quad (3.10)$$

where μ^2 is the scale where g^2 is defined, and c_1 is an arbitrary coefficient.

Next we consider the q^2 -dependence of $f^A(q^2, \rho)$ in (3.8). By definition (by choosing the coefficient c_1 in (3.10) correspondingly) it vanishes at $q^2 = 0$, and for large $q^2 \gg \rho$ the same scale anomaly arguments force it to behave, to one-loop order, as

$$f^A(q^2, \rho) \sim g^2 \beta_0 \ln \left(\frac{q^2}{\rho} \right) . \quad (3.11)$$

For our subsequent purposes it will be sufficient to replace the logarithmic increase in (3.11) by a positive constant

for $q^2 \rightarrow \infty$, since momenta with $q^2 \gg \rho$ will be cut off anyhow (see below). Thus we parameterize f^A as

$$f^A(q^2, \rho) \sim \frac{a_1 q^2}{a_2 \rho + q^2}, \quad (3.12)$$

with a_1, a_2 positive numerical coefficients of $\mathcal{O}(g^2)$, $\mathcal{O}(1)$, respectively. Now the kinetic terms in (3.8) are of the form of the kinetic terms of the model in Sect. 2, provided we identify Z_0^A in (2.3) with $Z_0^A(\rho, \mu^2)$ in (3.8) (or (3.10)), and Λ^2 in the denominator in (2.3) with ρ in the denominator in (3.12).

Next we will discuss the leading perturbatively irrelevant terms in $\Gamma_{\text{eff}}(A_\mu, \rho)$ generated by loops of the (massive) charged fields. Again, by $U(1)$ invariance (and a discrete Z_2 -symmetry $A_\mu \rightarrow -A_\mu$), these have to be quartic in the abelian field strength $\sim (F_{\mu\nu})^4$, convoluted with a form factor of the four in- or out-going momenta. Dimensional analysis dictates that, for large and equal Euclidean momenta q^2 , the form factor has to decay like q^{-4} . For distinct momenta this form factor will be a complicated function including possible open Lorentz indices.

For our subsequent purposes it will be sufficient to assume the presence of a particularly simple structure among all possible terms $\sim (F_{\mu\nu})^4$, which we parameterize as

$$\frac{\lambda^2}{8} \int d^4x \mathcal{O}(x) \mathcal{O}(x), \quad (3.13a)$$

$$\begin{aligned} \mathcal{O}(x) = & \int \mathcal{D}q e^{iqx} \int \mathcal{D}p F_{\mu\nu}(p+q) h(p^2) \\ & \times F_{\mu\nu}(q-p), \end{aligned} \quad (3.13b)$$

where, for the above reasons, $h(p^2)$ has to decay like p^{-2} for large p^2 . λ^2 is of the order of

$$\lambda^2 = \frac{\widehat{\lambda}^2 g^4}{16\pi^2 \rho^2}, \quad (3.14)$$

where $\widehat{\lambda}$ is of $\mathcal{O}(1)$.

Note that the expression (3.13a) appears with a positive sign. This follows from the limit of large field strengths $F_{\mu\nu}$ (at vanishing momenta), where the dependence of the induced effective action on $F_{\mu\nu}^2$ must be of the form

$$\int d^4x F_{\mu\nu}^2 \left(1 + g^2 \beta_0 \ln \left(\frac{C_1 + F_{\mu\nu}^2}{C_2} \right) \right), \quad (3.15)$$

for some constants $C_1 \sim C_2 \sim \Lambda_{\text{QCD}}^4$, in order to reproduce the scale anomaly [30]. Expanding (3.15) to $\mathcal{O}(F_{\mu\nu}^4)$ gives a positive coefficient.

Subsequently we replace $h(p^2)$ by a “sharp” cutoff,

$$h(p^2) \sim \theta(\rho - p^2). \quad (3.16)$$

Note that the scale of the “UV cutoff” in (3.16) has to be of $\mathcal{O}(\rho)$, since the contribution (3.13a) to $\Gamma_{\text{eff}}(A_\mu, \rho)$ was generated by loops of the charged fields with masses of $\mathcal{O}(\rho)$ and consequently $h(p^2)$ decays only for $p^2 \gg \rho$.

With the sharp cutoff we throw away logarithmic effective two-loop divergences, which would contribute to the A_μ propagator (and hence to the renormalization of g^2) once one-loop diagrams with the effective vertex (3.13) are computed. Here, however, we are not interested in the two-loop β -function, but in capturing the essential features of the infrared regime.

Hence, in the present approximation, $\Gamma_{\text{eff}}(A_\mu, \rho)$ coincides with $S(A_\mu)$ of the previous model in (2.1), provided we perform the following replacements of the parameters of the model in Sect. 2:

- (i) replace $Z^A(q^2)$ in (2.1) and (2.3) by $Z^A(q^2, \rho)$, with

$$Z^A(q^2, \rho) = Z_0^A(\rho, \mu^2) + \frac{a_1 q^2}{a_2 \rho + q^2}, \quad (3.17)$$

where, to one-loop order, $Z_0^A(\rho, \mu^2)$ has the form given in (3.10);

- (ii) replace Λ^2 in the θ -function in (2.6) by ρ , and λ by a ρ -dependent expression of the form (3.14).

At this point it may be helpful to summarize the procedure and the approximations under which the Yang–Mills theory turns into the confining abelian model of Sect. 2:

(1) An auxiliary field ρ for a dimension 2 condensate is introduced, such that all charged gauge fields and ghosts are massive for $\langle \rho \rangle \neq 0$ (which remains to be shown).

(2) The path integral over the charged gauge fields and ghosts, as well as over the “high momentum modes” (with $p^2 > \Lambda^2$) of the neutral gauge field A_μ , is performed.

(3) The resulting effective action $\Gamma_{\text{eff}}(A_\mu, \rho)$ is not computed exactly, but assumed to be well approximated by the following form:

(a) The term quadratic in A_μ in Γ_{eff} is given by the wave function normalization function $Z^A(q^2, \rho)$ of (3.17), whose dependence on q^2 and ρ is known (from the scale anomaly) for large q^2 or ρ , but parameterized by an “educated guess” for small q^2 and ρ . For vanishing q^2 , $Z^A(0, \rho) \equiv Z_0^A(\rho, \mu^2)$ is assumed to be given by an extrapolation of the one-loop (or scale anomaly) result (3.10), valid for large ρ , towards small ρ . Then, notably, $Z^A(0, \rho)$ turns negative for $\rho < \mathcal{O}(\Lambda_{\text{QCD}}^2)$. This (and only this) assumption is crucial for the subsequent results.

The dependence of $Z^A(q^2, \rho)$ on q^2 has to be interpolated between $Z^A(0, \rho)$ and the known scale anomaly result (3.11) for large q^2 . Our subsequent qualitative results do not depend on the precise dependence of $Z^A(q^2, \rho)$ on q^2 ; therefore, in order to allow for an analytic computation of the integral over q^2 appearing below, we parameterize its q^2 -dependence by the simple analytic structure (3.17), which replaces the logarithmic increase for $q^2 \rightarrow \infty$ by a constant a_1 .

(b) The term quartic in A_μ in Γ_{eff} is approximated by the F^4 term described in (3.13)–(3.16), i.e. more complicated tensorial structures are dropped and the form factor is replaced by the sharp cutoff (3.16). As in the case of the simplified parameterization of the q^2 -dependence of $Z^A(q^2, \rho)$ above, these approximations do not qualitatively affect the

$$e^{-G(J)} = \int \mathcal{D}A_{<A^2} \mathcal{D}\rho \mathcal{D}\phi e^{-\int \mathcal{D}q \{ (1/4)F_{\mu\nu} Z_{\text{eff}}^A(\phi, q^2, \rho) F_{\mu\nu} + (1/(2g^2))\rho^2 + \widehat{V}(\rho) - (1/8)\phi^2 + (1/(2\beta))(q_\mu A_\mu)^2 - J_\mu A_\mu \} } , \quad (3.18)$$

results below, but allow for subsequent analytic computations. Finally, terms of higher order in A_μ are dropped in Γ_{eff} , again for computational simplicity.

The Yang–Mills partition function (3.7) can then be rewritten, as in the previous model, invoking an auxiliary field ϕ . Including a source for A_μ as in (2.4) one obtains (see (3.18) on top of the page) with

$$Z_{\text{eff}}^A(\phi, q^2, \rho) = Z^A(q^2, \rho) + \lambda\phi\theta(\rho - q^2) . \quad (3.19)$$

The term $\rho^2/2g^2$ in the exponent in (3.18) originates from \mathcal{L}_ρ (3.5), and $\widehat{V}(\rho)$ from the path integrals over W_μ^\pm and c^\pm . In [7–10] one-loop expressions for $\widehat{V}(\rho)$ have been used in order to fix the VEV $\widehat{\rho}$ of ρ (the saddle point of the ρ path integral), with the unsatisfactory result that $\widehat{\rho}$ depends correctly on Λ_{QCD} only for a fine-tuned value of the gauge parameter α . Here we argue, instead, that $\widehat{\rho}$ is determined by another contribution to $V(\rho)$, which is obtained in analogy to the model in Sect. 2. Then $\widehat{\rho}$ depends automatically on Λ_{QCD} , as it should (see below).

The full effective potential $V_{\text{eff}}(\phi, \rho)$ is obtained by performing the A_μ path integral in (3.18), i.e.

$$V_{\text{eff}}(\phi, \rho) = \frac{1}{2g^2}\rho^2 + \widehat{V}(\rho) - \frac{1}{8}\phi^2 + \Delta V(\phi, \rho) , \quad (3.20)$$

with $\Delta V(\phi, \rho)$ as in (2.8):

$$\Delta V(\phi, \rho) = \frac{3}{2} \int_{k^2}^{\Lambda^2} \frac{q^2 dq^2}{16\pi^2} \ln(Z_{\text{eff}}^A(\phi, q^2, \rho)) . \quad (3.21)$$

If we replace the upper limit Λ^2 of the q^2 integral in (3.21) by ρ , the result for $\Delta V(\phi, \rho)$ can be obtained from the previous results in Sect. 2 after simple substitutions. The “error” is then given by a q^2 integral ranging from ρ to Λ^2 . However, below we are only interested in $\Delta V(\phi, \rho)$ near the saddle point $\widehat{\rho}$; choosing, at the end, $\Lambda^2 \sim \widehat{\rho}$, this error can be made arbitrarily small.

The ϕ -dependent terms in $\Delta V(\phi, \rho)$ are then given by $V(\phi)$ as in (2.11), after replacing Λ^2 by ρ everywhere, and $Z_{\text{eff}}^A(\phi, 0)$ by

$$Z_{\text{eff}}^A(\phi, 0, \rho) = Z_0^A(\rho, \mu^2) + \lambda\phi . \quad (3.22)$$

The ϕ -independent terms in $\Delta V(\phi, \rho)$, neglected in (2.11), are quadratic in ρ and of $\mathcal{O}(\rho^2/16\pi^2)$, hence negligible compared to the “tree level” term $\rho^2/2g^2$ in (3.20).

In order to determine the extrema $\widehat{\phi}$, $\widehat{\rho}$ of $V_{\text{eff}}(\phi, \rho)$ we first look for extrema with respect to ϕ , plug the resulting expression $\widehat{\phi}(\rho)$ back into $V_{\text{eff}}(\widehat{\phi}(\rho), \rho)$, and minimize with respect to ρ at the end.

The equation for extrema with respect to ϕ can again be taken from Sect. 2, see (2.12), with the substitutions above. Hence we obtain the following important results:

(i) again a confining saddle point exists, which shows the behavior (2.13) and, in analogy with (2.15),

$$Z_{\text{eff}}^A(\widehat{\phi}, 0, \rho) = Z_0^A(\rho, \mu^2) + \lambda\widehat{\phi} = 0 . \quad (3.23)$$

Equation (3.23) fixes the dependence of $\widehat{\phi}(\rho)$ on ρ :

$$\widehat{\phi}(\rho) = -\lambda^{-1} Z_0^A(\rho, \mu^2) . \quad (3.24)$$

(Recall that here λ depends on ρ , cf. (3.14).)

(ii) the necessary condition for the confining phase to exist, the analog of (2.16) with λ^2 as in (3.14), now reads

$$Z_0^A(\rho, \mu^2) < -\mathcal{O}\left(\left(\frac{g^2}{8\pi^2}\right)^2\right) , \quad (3.25)$$

i.e. essentially

$$Z_0^A(\rho, \mu^2) < 0 . \quad (3.26)$$

It remains to show that a saddle point $\widehat{\rho}$ with the above properties actually exists. Neglecting in V_{eff} terms of $\mathcal{O}(\rho^2/16\pi^2)$ relative to $\rho^2/2g^2$ as above, $\Delta V_{\text{eff}}(\widehat{\phi}(\rho), \rho)$ can be dropped and $V_{\text{eff}}(\widehat{\phi}(\rho), \rho)$ is simply given by

$$\begin{aligned} V_{\text{eff}}(\widehat{\phi}(\rho), \rho) &= \frac{1}{2g^2}\rho^2 - \frac{1}{8}\widehat{\phi}(\rho)^2 + \widehat{V}(\rho) \\ &= \frac{1}{2g^2}\rho^2 - \frac{2\pi^2\rho^2}{\lambda^2 g^4} (Z_0^A(\rho, \mu^2))^2 + \widehat{V}(\rho) , \end{aligned} \quad (3.27)$$

where we have used (3.24) and (3.14). Let us first neglect $\widehat{V}(\rho)$ in (3.27), which was generated by the path integral over the charged fields: using the one-loop expression (3.10) for $Z_0^A(\rho, \mu^2)$ one finds that $V_{\text{eff}}(\widehat{\phi}(\rho), \rho)$ vanishes for $\rho \rightarrow 0$, and is negative for small ρ , since $Z_0^{A(1)}$ diverges logarithmically for $\rho \rightarrow 0$. Then there appears a minimum, where $Z_0^{A(1)}(\rho, \mu^2)$ is negative and of $\mathcal{O}(g^2)$. This minimum constitutes the desired confining saddle point $\widehat{\rho}$. If we continue to increase ρ , $V_{\text{eff}}(\widehat{\phi}(\rho), \rho)$ increases until it reaches a maximum where $Z_0^A(\rho, \mu^2)$ is positive (still of $\mathcal{O}(g^2)$), and for $\rho \rightarrow \infty$ it is unbounded from below as in the case of $V(\phi)$ for $\phi \rightarrow \infty$. Hence all we have to require from $\widehat{V}(\rho)$ is that it does not *destroy* the desired saddle point, i.e. that it is small enough for the corresponding value $\widehat{\rho}$.

$\widehat{V}(\rho)$, as computed in [7–10], is proportional to the arbitrary gauge parameter α , since it has its origin in the quartic ghost interaction term (the last term in (3.3)): with the definition (3.5) for ρ we have $\widehat{V}(\rho) \sim (\alpha/16\pi^2)\rho^2 \ln \rho^2$. Since α is multiplicatively renormalized [28], it is always possible to choose gauges (bare values of $\alpha < \mathcal{O}(1)$, since $\widehat{\lambda}$ in (3.27) is of $\mathcal{O}(1)$, and $Z_0^A(\widehat{\rho})$ is of $\mathcal{O}(g^2)$) such that the effect of $\widehat{V}(\rho)$ in (3.27) is negligible. The above

result is thus valid in abelian gauges with α smaller than some critical value of $\mathcal{O}(1)$.

From

$$Z_0^A(\hat{\rho}, \mu^2) \sim -\mathcal{O}(g^2) \quad (3.28)$$

one easily obtains, using the one-loop expression (3.10) for $Z_0^A(\hat{\rho}, \mu^2)$,

$$\hat{\rho} \sim c_1^{-1} \mu^2 e^{-(1/(g^2\beta_0))(1+\mathcal{O}(g^2))} \sim \Lambda_{\text{QCD}}^2, \quad (3.29)$$

which is the result announced above and which has to be contrasted with the results in [7–9], where the dimension 2 condensate scales correctly only for a particular choice of the gauge parameter α , $\alpha = (1/2)\beta_0$.

Actually, if one computes $Z_0^A(\hat{\rho}, \mu^2)$ to higher loop orders, (3.28) (or a more accurate minimization of the effective potential with respect to ρ) always *defines* a scale Λ_{QCD}^2 through its solution $\hat{\rho}$. From (3.24), with $\lambda \sim \rho^{-1}$, one finds immediately that also $\hat{\phi}(\hat{\rho}) \sim \Lambda_{\text{QCD}}^2$ and hence

$$\langle F_{\mu\nu} F_{\mu\nu} \rangle \sim \lambda^{-1} \hat{\phi} \sim \Lambda_{\text{QCD}}^4. \quad (3.30)$$

Clearly, in terms of the running coupling $g_{\text{R}}^2(\rho)$ defined in (3.9) through $Z_0^A(\hat{\rho}, \mu^2)$, the condition (3.25) for the confining saddle point corresponds to $g_{\text{R}}^2(\hat{\rho}) < 0$, i.e. g_{R}^2 has “passed” a Landau singularity. However, the kinetic term of the A_μ field in the full effective action in the presence of the condensate $\hat{\phi}$ is proportional to $Z_{\text{eff}}(\hat{\phi}, q^2, \hat{\rho})$ (cf. (3.19)) and is never negative, but vanishes for $q^2 \rightarrow 0$. Hence, if one insists on defining a running coupling $g(q^2)$ in terms of $Z_{\text{eff}}^A(\hat{\phi}, q^2, \hat{\rho})^{-1}$, it diverges in the confining phase for $q^2 \rightarrow 0$. We do not find this convention very appropriate, however, since it refers to the virtualities q^2 of the abelian gauge fields A_μ only. We prefer to continue to parameterize the vertices by the constant g^2 , keeping the q^{-4} behavior of the propagator $P_{\mu\nu}^A(q^2)$.

Let us discuss the couplings of the charged gauge fields to A_μ in more detail. The $U(1)$ gauge symmetry (left unbroken up to the standard $U(1)$ gauge fixing term) allows one to separate these couplings into two classes:

(i) couplings involving the abelian field strength $F_{\mu\nu}$. Here additional derivative(s) act on A_μ , which soften infrared divergencies of loops involving the A_μ propagator emerging from such vertices.

(ii) couplings involving the $U(1)$ covariant derivative $D_\mu = \partial_\mu \pm igA_\mu$ acting on charged fields. These appear in the $U(1)$ invariant kinetic energy terms for W_μ^\pm and the charged ghosts in the Lagrangian (3.2), (3.3). In the quantum effective action these kinetic terms appear multiplied with wave function renormalization constants Z_W, Z_c , respectively. In the following we will study the behavior of these constants in the infrared limit and find that they vanish; this automatically suppresses the couplings of the neutral to charged gauge fields in the infrared.

For a most general parameterization of the quantum effective action the wave function renormalization constants should actually be replaced by functions of the (covariant) Laplacian $D_\mu D_\mu$ or, in momentum space, by functions of q^2 plus the corresponding couplings to the neutral gauge

field A_μ required by $U(1)$ gauge invariance. In general, for large Euclidean non-exceptional momenta $q^2 \rightarrow \infty$, the parameters of the quantum effective action approach their “bare” values ($Z_W, Z_c = 1$). Subsequently we replace Z_W and Z_c by constants for simplicity, i.e. we compute these functions at $q^2 = 0$. Hence their vanishing does not suppress the associated couplings to A_μ completely, just the associated form factors in the limit $q^2 \rightarrow 0$.

In the simplified parameterization of the quantum effective action with constant Z_W, Z_c , the relevant terms (quadratic in W_μ^\pm and the charged ghosts) read

$$\begin{aligned} & \frac{Z_W}{2} (D_\mu W_\nu^+ - D_\nu W_\mu^-) (D_\mu W_\nu^- - D_\nu W_\mu^-) + \rho W_\mu^+ W_\mu^- \\ & + \frac{Z_W}{\alpha} (D_\mu W_\mu^+) (D_\nu W_\nu^-) \\ & + Z_c (D_\mu \bar{c}^+ D_\mu c^- + D_\mu \bar{c}^- D_\mu c^+) + \alpha \rho (\bar{c}^+ c^- + \bar{c}^- c^+). \end{aligned} \quad (3.31)$$

Here we have included the mass terms originating from \mathcal{L}_m in (3.6). To one-loop order Z_W and Z_c get renormalized only by “rainbow” diagrams where the rainbow corresponds to a A_μ propagator of the form (2.18) (we continue to work in the Landau gauge, $\beta \rightarrow 0$, for the abelian sector). For convenience we introduce the notation $\varphi = \{W_\mu^\pm, c^\pm\}$ in the following. Our aim is now to derive a renormalization group equation for Z_φ . From the $A_\mu\text{-}\varphi\text{-}\varphi$ vertices from (3.31) one finds that the one-loop contributions to Z_φ due to the rainbow diagrams are

$$\Delta Z_\varphi = c_\varphi g^2 Z_\varphi^2 \int \frac{q^2 dq^2}{16\pi^2} P_\varphi(q^2) P_A(q^2). \quad (3.32)$$

Here P_φ are the massive W_μ^\pm/c^\pm propagators,

$$P_\varphi(q^2) = \frac{1}{Z_\varphi q^2 + (\alpha)\hat{\rho}} \quad (3.33)$$

(where the factor α appears only for the ghosts, and equals 1 for W_μ^\pm), and $P_A(q^2)$ reads, from (2.18),

$$P_A(q^2) = \frac{a_2 \hat{\rho} + q^2}{a_1 q^4}. \quad (3.34)$$

The constants c_φ in (3.32) read (for $SU(2)$)

$$\begin{aligned} c_{\varphi=W} &= -\frac{1}{6}(17 - 3\alpha), \\ c_{\varphi=c} &= -3. \end{aligned} \quad (3.35)$$

Note that, for α not too large, we have $c_\varphi < 0$.

In order to derive a renormalization group equation from (3.32) we proceed as in the case of the computation of the effective potential $V(\rho, \phi)$. We introduce an infrared cutoff k^2 for the q^2 integral in (3.32), and replace the constants Z_φ by $Z_\varphi(k^2)$. Then we take the derivatives with respect to k^2 of both sides of (3.32) and study the running of $Z_\varphi(k^2)$. As before such an infrared cutoff could be implemented in the “Wilsonian” way by modifying the propagator $P_A(q^2)$, but for the present purposes it is sufficient to simply cut off the q^2 integral in (3.32) at its lower end.

(Also this procedure can be re-interpreted as a “sharp” Wilsonian cutoff function in $P_A(q^2)$.) After introduction of this infrared cutoff k^2 , and taking the derivative d/dk^2 on both sides of (3.32), one obtains

$$k^2 \frac{dZ_\varphi(k^2)}{dk^2} = -Z_\varphi^2(k^2) \frac{c_\varphi g^2}{16\pi^2} k^4 P_\varphi(k^2) P_A(k^2) , \quad (3.36)$$

which becomes in the deep infrared regime, $k^2 \ll \hat{\rho}$,

$$k^2 \frac{dZ_\varphi(k^2)}{dk^2} \cong -Z_\varphi^2(k^2) \frac{c_\varphi g^2}{16\pi^2} \frac{a_2}{(\alpha)} . \quad (3.37)$$

(Again the factor α in the denominator appears only for the charged ghosts.) Note that, if $P_A(q^2)$ would not behave as q^{-4} for $q^2 \rightarrow 0$, $Z_\varphi(k^2)$ would stop to run with k^2 for $k^2 \ll \hat{\rho}$. Equation (3.37) is easily solved with the result

$$Z_\varphi(k^2) = \frac{Z_\varphi(\Lambda^2)}{1 + Z_\varphi(\Lambda^2) \frac{c_\varphi g^2}{16\pi^2} \frac{a_2}{(\alpha)} \ln\left(\frac{k^2}{\Lambda^2}\right)} , \quad (3.38)$$

and hence, for $k^2 \rightarrow 0$ and with $c_\varphi < 0$, we obtain $Z_\varphi(0) = 0$ as announced. One can check that the contributions from multi-rainbow diagrams to the running of $Z_\varphi(k^2)$ are suppressed by higher powers of the bare coupling g^2 . Also, the renormalization of the ρ vertex to the charged fields (or their mass terms) is infrared finite precisely because of the massiveness of the charged fields, in spite of the q^{-4} behavior of the A_μ propagator.

The question arises, however, whether this suppression of the A_μ - φ - φ couplings for $q^2 \rightarrow 0$ does not invalidate the contributions of the φ -loops to the effective action $\Gamma_{\text{eff}}(A_\mu, \rho)$, which have been used extensively before. The essential features of these contributions, on the other hand, arise either from the virtualities q^2 of the φ -fields (W_μ^\pm, c^\pm) which are very large ($q^2 \rightarrow \infty$), or from $q^2 \sim \rho$ where the massiveness (infrared finiteness) of the charged propagators is used. These features remain valid even if the A_μ - φ - φ couplings become suppressed for $q^2 \ll \rho$, and Z_φ in the φ propagators becomes replaced by $Z_\varphi(q^2)$ with $Z_\varphi(0) = 0$.

This result completes the confinement criterion $Z_{\text{eff}} = 0$ of [14, 15], which now holds for all fields of pure Yang–Mills theory including the charged ones: the simple poles of their propagators disappear, since their mass terms remain finite. In the abelian gauge the A_μ - W_μ^+ - W_μ^- vertex form factor will of course not be symmetric in the three external momenta; the present result applies to the limit of vanishing momenta squared of the charged fields W_μ^\pm . Nevertheless this behavior of the vertex form factor helps to suppress infrared divergencies of higher order loop diagrams, which helps to render the present approach stable with respect to higher loop orders.

4 Conclusions and outlook

The aim of the present paper is to show how the cooperation of two condensates of dimension 2 and 4, respectively,

generates both the confinement condition $Z_{\text{eff}} = 0$ and a mass gap for the charged fields in the abelian gauge in the confining phase. The study of a subtle saddle point of the effective potential is required to this end, which is only visible in the limit where an artificial infrared cutoff goes to zero.

The only role of the dimension 2 condensate is actually to give masses to all charged fields, i.e. the charged gauge fields W_μ^\pm and the charged ghosts. We choose the BRST invariant combination of bilinear charged gauge fields and ghosts here; the ghost condensates in [7, 9] are claimed to induce masses for the charged gauge fields by loops and could, in principle, do the same job. The formal proof of gauge invariance of the Yang–Mills quantum effective action relies, however, on the vanishing of the expectation values of all BRST-exact operators. This is no longer guaranteed, if the BRST symmetry is spontaneously broken.

Here we have concentrated on the condition $Z_{\text{eff}} = 0$ for confinement, which corresponds to the absence of colored asymptotic states. It would be quite straightforward, however, to introduce additional (auxiliary) antisymmetric tensor fields $B_{\mu\nu}$ for the abelian field strength, and to study the corresponding effective action. As in the case of the $1/N$ -solvable abelian models [24] this would make the relation with monopole condensation and the area law for the Wilson loop explicit.

Also for simplicity we have insisted on simple parameterizations of the q^2 - and ρ -dependences of various terms in the effective action, in order to allow for an analytic study of the appearance of the confining saddle point. It would not be too hard to compute these dependencies exactly (to one-loop order); then, however, the confining saddle point induced by A_μ -loops could be studied only numerically and would be somewhat less obvious.

On the other hand our parameterizations reproduce the essential features of the relevant terms in the effective action, which allows one to study the essential mechanism behind the confining saddle point: without the condensate $\phi \sim F_{\mu\nu}^2, Z_{\text{eff}}^A(\hat{\rho})$ would turn negative for $\hat{\rho}$ small enough. This corresponds to a non-convexity of $\Gamma_{\text{eff}}(F_{\mu\nu}^2)$ around the origin, which is impossible. The condensate ϕ then renders $\Gamma_{\text{eff}}(F_{\mu\nu}^2)$ semi-convex, which corresponds to the non-analytic behavior of its effective potential.

The fact that these essential features are visible already after the computation of one-loop diagrams should not make one believe that confinement is “perturbative”: if one eliminates all auxiliary fields by its equations of motion at the very end it becomes clear that, by computing Γ_{eff} in its presence, one has implicitly summed up an infinite number of loops.

Nevertheless the question arises whether the present approach would allow for quantitatively stable higher order corrections, once lowest orders are computed with sufficient precision. (Given that even perturbation theory is not asymptotically stable, this is evidently a rather ambitious program.) More concretely, this corresponds to the question whether possible infrared divergencies from higher order corrections can be controlled or, better,

shown to be absent. Two steps in this direction are, in the present approach,

- (i) the massiveness of the charged gauge fields (and ghosts), and
- (ii) the vanishing of the wave function renormalization constants of the charged fields in the infrared. Notably this latter phenomenon will suppress very long range correlation functions between operators involving charged fields in spite of the q^{-4} behavior of the abelian propagator.

Finally we remark that the simultaneous presence of a mass gap (of the charged fields) and confining interactions (as induced by the abelian sector) can most likely be made explicit only in the abelian gauge. This gauge evidently plays an essential role in the present approach, which describes a quite explicit dynamical mechanism behind the confining phase in continuum Yang–Mills theory. An interesting task for the future will be the study of the constraints on the full effective action Γ_{eff} which arise from the Slavnov–Taylor identities in the abelian gauge (suitably generalized due to the presence of the auxiliary fields), once the present results on some selected terms in Γ_{eff} are taken into account.

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